On quantization of systems with second class constraints

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Abstract

It is shown that quantization of the dynamical systems with second class constraints actually can be reduced to quantization of the systems with first class constraints. The motion of the non-relativistic particle along the plane curve and on a surface is considered. The results coincide with those of the "thin layer method". Influence of the non-physical variables on the physical sector is demonstrated.

1 Introduction

The issue of dynamical systems with constraints dominates in the modern physics. Gauge theories describing all the known interactions (gravitational, electro-weak, strong) are typical examples of such systems. As it is well known [1], there are two classes of constraints: the first and the second ones. Appearance of constraints results in reduction of the phase space of the initial theory but mechanisms of the reduction differ for constraints from different classes. To elucidate the problem we can take the simplest case and consider the first class constraints as conditions on some canonical momenta $p_{r+1} = \ldots = p_{r+s} = 0$ (s first class constraints, $r+s=n$), and second class constraints — as conditions on momenta and the canonically conjugate coordinates $p_{r+1} = \ldots = p_{r+s} = 0$, $q_{r+1} = \ldots = q_{r+s} = 0$ ($2s$ second class constraints). In both cases dimension of the physical phase space (PS) becomes $2(n-s)$, because in the case of the first class constraints the variables $q_{r+1}, \ldots, q_{r+s}$ are also non-physical (they remain arbitrary). So, if we consider

theory with n degrees of freedom, then in both cases the dimension of the physical phase space is equal to $2n-2s$. In the classical theory this does not cause problems because for the systems with second class constraints there is no arbitrariness while for the systems with first class constraints the arbitrariness can be removed by gauge fixing

$$q_{r+1} = \ldots = q_{r+s} = 0. \tag{1}$$

It is admissible, because the theory does not impose any restrictions on evolution of the variables (1). Problems occur after transition to the quantum description. One cannot require the operator equations $\hat{p}_{r+1} \ldots = \hat{p}_{r+s} = 0$ to be fulfilled in the case of the first class constraints because it contradicts to commutation relations $[\hat{q}_{r+k}, \hat{p}_{r+l}] = i\hbar\delta_{kl}, 1 \leq k,l \leq s$. And of course, one cannot require it in presence of additional conditions $\hat{q}_{r+1} = \ldots = \hat{q}_{r+s} = 0$ (the second class constraints). In the former case there is natural way out [1]: it is sufficient to require realization of constraints on physical vectors Ψ_{ph}

$$\hat{p}_{r+1}\Psi_{ph} = \ldots = \hat{p}_{r+s}\Psi_{ph} = 0. \tag{2}$$

But this recipe does not suite for the second class constraints: conditions $\hat{p}\Psi_{ph} = \hat{q}\Psi_{ph} = 0$ are incompatible with canonical commutation relations $[\hat{q}, \hat{p}] = i\hbar$. Several ways out there was proposed. Two of them are based on the fact, that variables $q_{r+k}, p_{r+k}, 1 \leq k \leq s$, are non-physical. So, it is supposed that it is possible to change their dynamics arbitrary. Dirac [1] proposed to change the Poisson brackets $\{\Phi_i, \Phi_j\}$; for the second class constraints (i.e. $\Phi_i = 0$ $det\{\Phi_i, \Phi_j\} \neq 0$) he proposed to use new brackets

$$\{f,g\} \to \{f,g\}_D = \{f,g\} - \{f,\Phi_i\}(\{\Phi_i,\Phi_j\})^{-1}\{\Phi_j,g\}. \tag{3}$$

The Dirac brackets $\{f,g\}_D$ are equal to zero for constraints Φ_i, i.e. the latter become the first class constraints and it is possible to use conditions (2) $\hat{\Phi}_i\Psi_{Ph} = 0$. Thus, non-physical canonical conjugate variables q, p become independent (non-physical) variables. Variables canonically conjugated them are ignored [2, p. 130].

In the recipe [3–5] one doubles the number of non-physical variables; it is postulated that the Poisson brackets of all non-physical

canonical variables of the original system are equal to zero ("abelian conversion"), while new variables are assumed be canonically conjugated to them. Then, it is again possible to use recipe (2) for constraints Φ_i. In fact, the Dirac recipe is accomplished by introducing canonically conjugated partners for non-physical "momenta" Φ_i.

Restriction of motion in the configuration space (for example, on some hypersurface in Euclidean space defined by conditions $\varphi_i(q_1, \ldots, q_n) = 0, i = 1, \ldots, s < n$) is the typical reason for the second class constraints occurrence. In this approach quantum mechanics (QM) on a hypersurface can be considered as a limiting case of n—dimensional QM in an infinitesimally thin layer surrounding the hypersurface. Such a method of quantization in curved spaces is called "the thin layer method". It appears in two forms. In the first one it is required that the wave functions become zero on borders of the layer [6]. In this case decreasing of thickness of the layer to zero leads to occurrence of states with infinite energy, and "renormalization" of energy is needed for transition to QM on the hypersurface. In the second method [7] one introduces an oscillator potential in directions normal to the hypersurface; when the elasticity coefficient tends to infinity wave function turns out confined on the hypersurface. It was found in [7] that as a result some function V_q ("quantum potential") is added to the Beltrami—Laplace operator on the hypersurface (i.e. to kinetic energy operator). It turns out (and this is extremely important) that the potential V_q depends both on intrinsic and extrinsic curvatures of the hypersurface. By itself this fact is quite satisfactory, because wave function is a non-local object. It is remarkable that extrinsic curvature also influences motion of a quantum particle. In the classical mechanics motion of a point particle depends only on intrinsic geometry.

Path integral method gives quantum potential $V_q = \frac{\hbar^2 R}{12}$, where R — scalar curvature of space [8]. And only mechanics with the second class constraints allows to reveal the important fact that quantum potential depends not only on intrinsic but also on *extrinsic* curvature. In section 2 the results of different methods are listed for the simplest case — a particle on a sphere. All the mentioned methods give different results; it means that at least two of them are incorrect. In sec. 3 a new method of quantization, naturally following from rules of quantization of systems with the first

class constraints is presented. In sections 4, 5 it is shown, that for a plane curve and surfaces in 3-dimensional Euclidean space the method gives the same results as the thin layer method. In sec. 6 the influence of the non-physical sector of a system on the physical one is discussed. It is shown, that quantum potential depends on space, in which e.g. a curve is taken: a circle on a plane or on a sphere. Thus, in QM the non-physical sector influences the physical one.

2 Particle on a sphere — three recipes of quantization

In this section we present results of quantization by three various methods for simplest case — a particle on the sphere of radius R in R^n [5].

1. *The Dirac method.* Constraints:

$$\Phi_1 = \vec{x}^2 - R^2,$$

$$\Phi_2 = (\vec{p}, \vec{x}).$$

Hamiltonian:

$$\hat{H} = -\frac{\hbar^2}{2}\triangle_{n-1} + \frac{\hbar^2 n^2}{8R^2},$$

where $\triangle_{n-1}$ — the Beltramy—Laplace operator on the sphere. Quantum potential is:

$$V_q^D = +\frac{\hbar^2 n^2}{8R^2}. \tag{4}$$

2. *The Abelian conversion method.* Constraints Φ_1, Φ_2 are postulated abelian; new auxiliary variables Q and P, canonically conjugated to them are introduced.

$$\{Q, \Phi_1\} = \{P, \Phi_2\} = 1, \{Q, \Phi_2\} = \{P, \Phi_1\} = 0.$$

Then new constraints

$$\sigma_1 = \Phi_1 + P,$$

$$\sigma_2 = \Phi_2 + 2\vec{x}^2 Q,$$

are in involution:

$$\{\sigma_1, \sigma_2\} = 0.$$

The Hamiltonian is

$$H = \frac{1}{2(\sigma_1^2 + R^2)}(\sigma_2^2 + L_a^2),$$

where $L_a = x_i p_j - x_j p_i, \quad a \equiv (ij)$ — components of the angular momentum operator. Quantum potential is zero:

$$V_q^{AC} = 0.$$

3. *The thin layer method.* Motion on a surface is considered as a motion in the Euclidean space between two parallel (equidistant) surfaces when distance between the surfaces tends to zero [6]. The energy of the system tends in this case to infinity. More attractive looks the idea of taking "squeezing" potentials (e.g. by introducing an oscillator potential $V = \frac{1}{2}\gamma \vec{x}_\perp^2, \gamma \to \infty$, in normal directions) [7]. Then for a particle on a surface in 3-dimensional Euclidean space one obtains quantum potential

$$V_q^{ThL} = -\frac{\hbar^2}{2}(H^2 - K) = -\frac{\hbar^2}{8}(\frac{1}{R_1} - \frac{1}{R_2})^2,$$

where H — the mean curvature, K — the Gaussian curvature, R_1^{-1}, R_2^{-1} — the principal curvatures of the surface. For a motion on a circle ($R_1 = R, R_2 = \infty$) we have

$$V_q^{ThL} = -\frac{\hbar^2}{8R^2}.$$

For a motion on a sphere ($R_1 = R_2$) we have $V_q^{ThL} = 0$.

3 Reduction to the case of first class constraints

The following method naturally follows from the first class constraints quantization rules. Let $q_i, i = 1, \ldots, n,$ be curvilinear orthogonal coordinates in R^n so that conditions $q_{r+k} - C_k = 0$,

$C_k = const$, $k = 1, \ldots, s$, $r = n - s$, define a hypersurface with dimension $n - s$, i.e. coordinates $q_{r+1}, \ldots, q_n$ are normal to the hypersurface, and $p_{r+1}, \ldots, p_n$ are momenta canonically conjugated to them. Quantization can be done in two steps.

I. On solutions of the Schrödinger equation

$$-\frac{\hbar^2}{2} \triangle \Psi(q_1, \ldots, q_n) = E\Psi(q_1, \ldots, q_n), \tag{5}$$

we impose conditions

$$\hat{P}_{r+k}\Psi(q_1, \ldots, q_n) = 0, \quad k = 1, \ldots, n - s, \tag{6}$$

where $\triangle$ — the Laplace operator in R^n in curvilinear coordinates, and the momentum operators are:

$$\hat{P}_j = \frac{\hbar}{i} g^{-\frac{1}{4}} \hat{\partial}_j g^{\frac{1}{4}}, \quad \hat{P}_j^+ = \hat{P}_j, \quad j = 1, \ldots, n, \tag{7}$$

$\hat{P}_j$ are Hermitean in the Hilbert space with the inner product $(f_1, f_2) = \int d^n q \sqrt{g} \bar{f}_1(q) f_2(q)$ where g is determinant of the metric tensor. After imposing conditions (6) the wave functions actually do not depend on non-physical coordinates

$$\Psi_{ph}(q_1, \ldots, q_n) = g^{-\frac{1}{4}} \Phi(q_1, \ldots, q_r),$$

and it is possible to consider the latter not as dynamical variables but as parameters, because now in the normalization condition one integrates only on physical variables

$$\int d^r q \sqrt{g} \mid \Psi_{ph} \mid^2 = 1. \tag{8}$$

II. Further, it is necessary: 1) substitute in Jacobians $\sqrt{g}$ all q_{r+k}, $k = 1, 2, \ldots, s$ by C_k (resolve constraints $q_{r+k} - C_k = 0$); 2) move in the Hamiltonian all operators of non-physical momenta $\hat{P}_{r+k}$ to the right (taking into account the commutation relationes) and put them equal to zero (see (6)); 3) replace in the final operator all q_{r+k} by C_k. It turns out, that this method gives the same results as the thin layer method. The method looks natural, because in the case of the first class constraints sometimes "additional conditions" (such as conditions (1)), are introduced ("gauge fixing").

In the following sections we shall see, that this recipe in the natural way gives the Schrödinger equation on hypersurface with quantum potential depending both on intrinsic (scalar curvature) and extrinsic geometry (curvature of plane curve in R^2 and mean curvature H of a surface in R^3). This quantum potential depends not only on geometry of hypersurface, but also on geometry of imbedding space as shows an example of a circle in R^2 and on a sphere S^2 (section 6). Notice, that we do not introduce squeezing potentials, as in [6, 7], which should keep a particle on the hypersurface. The idea of this "two-step reduction" recipe was introduced in [9]. The recipe (5), (6) was considered [10], but was rejected in favor of the abelian conversion method.

4 Quantum potential on a plane curve.

We shall illustrate the recipe (5), (6) first for a plane curve with curvature $k(s)$, where s is length of an arc. We use special curvilinear coordinates ("coordinates of a thin layer ") q_1, q_2, where q_1 — length of the arc of a curve from some "zero" point, q_2 — distance between the curve and a plane point. The first quadratic form in such (semi-geodesic) coordinates becomes

$$dl^2 = dq_2^2 + g dq_1^2,$$

where

$$g = [1 - q_2 k(q_1)]^2.$$

The lines $q_2 = const$ are equidistant curves parallel to the original curve. Lines $q_1 = const$ are normal to it. The Schrödinger equation for a free particle on a plane reads

$$-\frac{\hbar^2}{2}\triangle\Psi(q_1, q_2) = E\Psi(q_1, q_2), \tag{9}$$

and the condition on the wave function is

$$\hat{P}_2\Psi(q_1, q_2) = 0, \tag{10}$$

where the momentum operator $\hat{P}_2$ is given by equation (7).

It follows from (7), (10) that the wave function has the form:

$$\Psi(q_1, q_2) = g^{-\frac{1}{4}}\Phi(q_1). \tag{11}$$

Substituting this expression in (9) we obtain

$$-\frac{\hbar^2}{2}\left(g^{-\frac{1}{2}}\partial_1[g^{\frac{1}{2}}g^{-1}\partial_1(g^{-\frac{1}{4}}\Phi)] - (g^{-\frac{1}{4}}\Phi)[g^{-\frac{1}{4}}\partial_2^2 g^{\frac{1}{4}}]\right) = E(g^{-\frac{1}{4}}\Phi),$$

$$g^{-\frac{1}{4}}\partial_2^2 g^{\frac{1}{4}} = -\frac{1}{4}\frac{k^2}{(1-q_2k)^2}, \tag{12}$$

Taking $q_2 = 0$, we reproduce the result of papers [6, 7]:

$$-\frac{\hbar^2}{2}\partial_1^2\Phi - \frac{\hbar^2}{8}k^2\Phi = E\Phi,$$

i.e. quantum potential is equal to

$$V_q = -\frac{\hbar^2}{8}k^2. \tag{13}$$

5 Quantum potential for a particle on a surface in three-dimensional space

Let's consider a surface Γ in space R^3. We introduce coordinates q_1, q_2 such that the first and the second quadratic forms become diagonal

$$ds^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2.$$

Following authors [6, 7], we introduce in the thin layer special curvilinear orthogonal coordinates (coordinates of a thin layer). Equation $\vec{r} = \vec{r}(q_1, q_2)$ parametrizes the surface. A point in R^3 is characterized by coordinates (q_1, q_2) of the surface Γ and by the distance q_3 from it:

$$\vec{R}(q_1, q_2, q_3) = \vec{r}(q_1, q_2) + q_3\vec{n}(q_1, q_2),$$

where $\vec{n}$ is a normal to the surface. Then the metrics in R^3 is:

$$ds^2 = H_1^2 dq_1^2 + H_2^2 dq_2^2 + H_3^2 dq_3^2,$$

where

$$H_1 = h_1(1 - q_3 k_1), \quad H_2 = h_2(1 - q_3 k_2), \quad H_3 = 1$$

and k_1, k_2 are principal curvatures.

The metrics of the surfaces $q_3 = const$, parallel to the surface Γ, is defined by

$$ds^2 = H_1^2 dq_1^2 + H_2^2 dq_2^2.$$

The stationary Schrödinger equation for a free particle in R^3 reads

$$-\frac{\hbar^2}{2} \triangle_3 \Psi(q_1, q_2, q_3) = E\Psi(q_1, q_2, q_3).$$

We demand

$$\hat{P}_3 \Psi = 0,$$

where $\hat{P}_3$

$$\hat{P}_3 = \frac{\hbar}{i} G^{-\frac{1}{4}} \hat{\partial}_3 G^{\frac{1}{4}},$$

$$G^{\frac{1}{2}} = H_1 H_2 H_3 = h_1 h_2 (1 + q_3(k_1 + k_2) + k_1 k_2 q_3^2) = h_1 h_2 (1 + 2q_3 H + K q_3^2),$$

where K and H are the Gauss and mean curvatures of the surface Γ. The wave function Ψ and the Laplace operator become

$$\Psi(q_1, q_2, q_3) = G^{-\frac{1}{4}} \Phi(q_1, q_2),$$

$$\triangle_3 = \triangle_2 + G^{-\frac{1}{2}} \hat{\partial}_3 (G^{\frac{1}{2}} \hat{\partial}_3),$$

where $\triangle_2$ — the Beltrami—Laplace operator on the surfaces $q_3 = const$, i.e.

$$\triangle_3 \Psi = \triangle_2 \Psi + \Psi G^{-\frac{1}{4}} \partial_3 (G^{\frac{1}{2}} \partial_3 G^{-\frac{1}{4}}).$$

Taking $q_3 = 0$ we reproduce the result of papers [6, 7] :

$$-\frac{\hbar^2}{2} \triangle_2 \Psi(q_1, q_2) - \frac{\hbar^2}{2} (H^2 - K) \Psi(q_1, q_2) = E\Psi(q_1, q_2),$$

i.e. quantum potential is equal to :

$$V_q = -\frac{\hbar^2}{2} (H^2 - K). \tag{14}$$

6 Quantum potential and geometry of embedding space

As it was noticed in Introduction quantum potential depends on geometry of imbedding space. To demonstrate this we consider a circle on a sphere and on a plane.

On the sphere of radius R take a circle $\theta = const$ (θ,ϕ are spherical coordinates). Here ϕ is the physical coordinate, while θ — the non-physical one. Metrics is given by $ds^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2)$, and $g = R^4 \sin^2\theta$. According to Eq. (6) physical states are: $\Psi(\phi,\theta) = g^{-\frac{1}{4}}\Phi(\phi) = R^{-1}(\sin\theta)^{-\frac{1}{2}}\Phi(\phi)$. The Schrödinger equation (5) takes form:

$$-\frac{\hbar^2}{2}\left(R^{-2}\sin^{-2}\theta\frac{\partial^2\Psi}{\partial\theta^2} + \Psi R^{-2}\sin^{-\frac{1}{2}}\theta\frac{\partial}{\partial\theta}[\sin\theta\frac{\partial}{\partial\theta}\sin^{-\frac{1}{2}}\theta]\right) = E\Psi$$

and quantum potential is:

$$V_q = (-\frac{\hbar^2}{2})R^{-2}\sin^{-\frac{1}{2}}\theta\frac{\partial}{\partial\theta}[\sin\theta\frac{\partial}{\partial\theta}\sin^{-\frac{1}{2}}\theta] = -\frac{\hbar^2}{8R^2}[1 + \frac{1}{\sin^2\theta}]. \tag{15}$$

On the other hand, in the case of the circle with $R_c = R\sin\theta$ on a plane we had

$$V_q = -\frac{\hbar^2}{8(R\sin\theta)^2} = -\frac{\hbar^2}{8R_c^2}. \tag{16}$$

We see that quantum potential depends on the curvature of the imbedding space. Of course in the limit $R \to \infty, \theta \to 0, R\sin\theta \to R_c$ the results (15) and (16) coincide. In the framework of QM the difference between Eqs. (15) and (16) is both natural and desirable: wave function is a non-local object and we began with formulation of the problem in the imbedding space.

7 Conclusion

The analysis of existing quantization methods of systems with second class constraints shows that this problem is less trivial than similar problem in the case of first class constraints. These methods of quantization can be divided into two groups.

To the first group we attribute the methods admitting that the non-physical sector cannot influence the physical one, so one can arbitrary change dynamics of the non-physical variables: for example to replace their Poisson brackets by Dirac brackets [1] or to set Poisson brackets equal to zero for all non-physical canonical variables [3, 4], adding extra non-physical variables. Application of these recipes to the elementary systems gives the different results that means: in the quantum theory one cannot change arbitrary the dynamics of the non-physical sector.

The methods in the second group do not modify dynamics of the non-physical sector. So, in papers [6, 7] the quantization is made in space of all variables, both physical and non-physical ones ("the thin layer method"). In the limit of vanishing thickness of the "layer" one gets quantum theory on the physical subspace. The final result differs from results, received by methods from the first group. Unfortunately this recipe requires large auxiliary work. It turns out however that there is a more direct way — method of "reduction the problem to the first class constraints problem" resulted in sec. 3.

Let's address in conclusion the question of influence of non-physical sector on the physical one. We see, that the non-physical sector influences the physical one in all the methods of quantization. It means that the problem of quantization on, say, a curve is by itself set incorrectly. It is necessary to specify space, in which the curve is imbedded. For example, in the case of a plane, the quantum potential V_q is given by (16) (radius of a circle is equal to $R\sin\theta$), and if the circle is on the sphere of radius R the potential V_q is given by (15). The result looks paradoxical only from the point of view of the classical theory. In quantum theory the motion of particles is described by wave functions, and it is not surprising that the motion on a circle depends on the outer space — sphere or plane. In the first case the wave function does not depend on one of the spherical coordinates (angle θ), while in second one — on the radial variable.

Conclusion: in quantum mechanics description of motion in curved spaces by itself, i.e. ignoring the imbedding space, is senseless.

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